

**Correction to the paper
“Some remarks on Davie’s uniqueness theorem”**

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Abstract

The property 4 in Proposition 2.3 from the paper “Some remarks on Davie’s uniqueness theorem” is replaced with a weaker assertion which is sufficient for the proof of the main results. Technical details and improvements are given.

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1. INTRODUCTION

We consider the stochastic differential equation

$$X_t = x + W_t + \int_0^t b(s, X_s) ds. \quad (1)$$

In the paper [1] the following theorem was proved:

Theorem 1.1. *Let $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a Borel measurable bounded mapping. Then for almost all Brownian paths the equation 1 has exactly one solution.*

In the work [11] an alternative approach was proposed. However as it was pointed out in [10] (see Remark 5.3, p. 24) the uniform Hölder continuity (the property 4 from Proposition 2.3 in [11]) doesn’t immediately follow from Kolmogorov continuity theorem and the moments estimates established in [11]. Below we present a simple modification of Kolmogorov continuity theorem and adjust the proofs of the main results from [11] accordingly. Some other observations regarding the regularity of the flow, in particular, a simple treatment of the case of a bounded drift, are not included into this short note and will be discussed in a separate paper.

2. AUXILIARY RESULTS

Proposition 2.3. *Let*

$$b \in L^q([0, T], L^p(\mathbb{R}^d)), \quad \frac{d}{p} + \frac{2}{q} < 1.$$

Then, there exists a Hölder flow of solutions to the equation 1. More precisely, for any filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and a Brownian motion W , there exists a mapping $(s, t, x, \omega) \mapsto \varphi_{s,t}(x)(\omega)$ with values in \mathbb{R}^d , defined for $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, such that for each $s \in [0, T]$ the following conditions hold:

- 1. for any $x \in \mathbb{R}^d$ the process $X_{s,t}^x = \varphi_{s,t}(x)$ is a continuous $\mathcal{F}_{s,t}$ adapted solution to the equation 1,*
- 2. P -almost surely the mapping $x \mapsto \varphi_{s,t}(x)$ is a homeomorphism,*
- 3. P -almost surely for all $x \in \mathbb{R}^d$ and $0 \leq s \leq u \leq t \leq T$*

$$\varphi_{s,t}(x) = \varphi_{u,t}(\varphi_{s,u}(x)),$$

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4. For any $\alpha \in (0, 1)$, $\eta > 0$, $N > 0$ and a given increasing sequence S of finite sets $\{S_n\}_{n=0}^\infty$ with $|S_n| \leq 2^n$ there exists a set Ω' of probability 1 such that for any $s \in S_n$ $x, y \in \mathbb{R}^d$ with $|x|, |y| < N$, $|x - y| \leq 2^{-n}$ and each $t \in [s, T]$

$$|\varphi_{s,t}(x) - \varphi_{s,t}(y)| \leq C(\alpha, T, N, S, \omega) |x - y|^\alpha.$$

Following the proof given in [11] we consider the process

$$Y_t := \psi_t(t, X_t) = X_t + U(t, X_t)$$

which is the unique solution of the transformed equation

$$dY_t = \tilde{b}(t, Y_t) dt + \tilde{\sigma}(t, Y_t) dW_t,$$

for details see [11]. In the work [11] the following bound was established:

$$\mathbb{E} \sup_{t \in [0, T]} |Y_t^x - Y_t^y|^a \leq C(a, T) (|x - y|^a + |x - y|^{a-1}), \quad (2)$$

It is easy to see that the same arguments provide the estimate

$$\sup_{s \in [0, T]} \mathbb{E} \sup_{t \in [s, T]} |Y_{s,t}^x - Y_{s,t}^y|^a \leq C(a, T) (|x - y|^a + |x - y|^{a-1})$$

Since ψ_t, ψ_t^{-1} are Lipschitz continuous uniformly in time an analogous bound holds for $X_{s,t}^x$

$$\sup_{s \in [0, T]} \mathbb{E} \sup_{t \in [s, T]} |X_{s,t}^x - X_{s,t}^y|^a \leq C(a, T) (|x - y|^a + |x - y|^{a-1})$$

We can assume (see [2]) that for each s the mapping $X_{s,t}^x$ is jointly continuous with respect to t, x . To complete the proof we will need the following lemma:

Lemma 2.1. *Let $X(s, x)$ be a continuous with respect to x process with values in a complete metric space (M, ϱ_M) on $\mathcal{S} \times [0, 1]^d$. Assume that for some $a, b > 0$*

$$\sup_{s \in \mathcal{S}} \mathbb{E} \varrho_M(X_s(u), X_s(v))^a \leq |u - v|^{d+b}, \quad u, v \in [0, 1]^d$$

For any $\alpha \in (0, b/a), \eta \in (0, b - \alpha a)$ and any increasing sequence S of finite subsets $\{S_n\}_{n=0}^\infty$ with $|S_n| \leq 2^n$ there exists a set Ω' of probability 1 such that

$$\varrho_M(X_s(u), X_s(v)) \leq C(\alpha, \eta, S, \omega) |u - v|^\alpha \quad s \in S_n, u, v \in [0, 1]^d, |u - v| \leq 2^{-n}, \omega \in \Omega',$$

The proof is a minor modification of the standard proof of Kolmogorov continuity theorem, for details see [9].

Proof. Let $\alpha \in (0, b/a)$. Define D_n as

$$D_n := \left\{ (k_1, \dots, k_d) 2^{-n}; k_1, \dots, k_d \in \{1, \dots, 2^n\} \right\}$$

Let

$$Y(s, n) := \max \left\{ \varrho_M(X_s(u), X_s(v)); u, v \in D_n, |u - v| = 2^{-n} \right\}$$

Then

$$\mathbb{E} (2^{\alpha n} Y(s, n))^a \leq C 2^{\alpha a n} 2^{d n} (2^{-n})^{d+b} \leq C 2^{(\alpha a - b)n}$$

Now one readily sees that

$$\mathbb{E} \sum_{n=1}^{\infty} \sum_{s \in S_n} (2^{\alpha n} Y(s, n))^a < \infty$$

Consequently, there exists a set Ω' of full measure such that

$$\sum_{n=1}^{\infty} \sum_{s \in S_n} (2^{\alpha n} Y(s, n))^a < C(\omega) < \infty, \omega \in \Omega'.$$

in particular

$$Y(s, n)(\omega) \leq C'(\omega) 2^{-\alpha n}, \quad s \in S_n, \omega \in \Omega'$$

Using the fact that the sequence S is increasing we obtain the bound

$$Y(s, m)(\omega) \leq C'(\omega) 2^{-\alpha m}, \quad s \in S_n, m \geq n, \omega \in \Omega'$$

Now let s be a fixed point in S_n . Applying the standard arguments one can see that for each $m \geq n$ and any $u, v \in D_m$ such that $|u - v| \leq 2^{-n}$ the following inequality holds:

$$\varrho_M(X_s(u), X_s(v)) \leq C''(S, \omega) |u - v|^\alpha$$

Now it is easy to complete the proof. \square

Now let us come back to the proof of the property 4. Define a random mapping J from $[0, T] \times [-N, N]^d$ to the Banach space $C([0, T], \mathbb{R}^d)$ equipped with the standard sup-norm as follows:

$$J(\omega, s, x)(t) := X_{s, \min(s+t, T)}^x(\omega).$$

The joint continuity of $X_{s,t}^x$ with respect to t, x immediately implies the mapping J is continuous. Next, the estimate

$$\sup_{s \in [0, T]} \mathbb{E} \sup_{t \in [s, T]} |X_{s,t}^x - X_{s,t}^y|^a \leq C(a, T) (|x - y|^a + |x - y|^{a-1})$$

can be written as

$$\sup_{s \in [0, T]} \mathbb{E} \|J(s, x) - J(s, y)\|^a \leq C(a, T) (|x - y|^a + |x - y|^{a-1})$$

For any $\alpha \in (0, 1)$ and $\eta > 0$ one can find sufficiently large $a > 0$ such that

$$\alpha < \frac{a - 1 - d}{a}, \quad \eta < a - 1 - d - \alpha a$$

so now it is easy to complete the proof applying Lemma 2.1.

3. MAIN RESULTS

In this section we adjust the proofs of the main results stated in the paper [11] using the corrected version of the property 4 from Proposition 2.3.

Theorem 3.1. *Assume that the coefficient b satisfies the following conditions:*

1. *there exists $M_1 \in L^{q_1}([0, T], \mathbb{R})$ such that*

$$|b(t, x)| \leq M_1(t), \quad t \in [0, T], \quad x \in \mathbb{R}^d$$

2. *there exists $M_2 \in L^{q_2}([0, T], \mathbb{R})$ and $\beta > 0$ such that*

$$|b(t, x) - b(t, y)| \leq M_2(t) |x - y|^\beta, \quad t \in [0, T], \quad x, y \in \mathbb{R}^d$$

3. *one has*

$$q_1 \geq q_2 > 2, \quad \beta > 0, \quad \frac{\beta}{p_1} + \frac{1}{p_2} > 1, \quad \text{where } \frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Then there exist a set Ω' with $P(\Omega') = 1$ such that for each $\omega \in \Omega'$ the equation 1 has exactly one solution.

Proof. Let Y_t be a solution to the equation 1 for a fixed Brownian trajectory W . Then the following estimate holds:

$$\max_{t \in [0, T]} |Y_t| \leq |x| + \max_{t \in [0, T]} |W_t| + T^{\frac{1}{p_1}} \|M_1\|_{L^{q_1}[0, T]} =: M(x, W),$$

so without loss of generality we can assume that $b(t, x) = b(t, x)I_{\{|x| < N\}}$ for some $N > 0$. Then Proposition 2.3 (it is clear that one can take q_1 for q and any sufficiently large positive number for p) yields that P -almost surely the equation 1 has a Hölder-continuous flow of solutions which will be denoted by $X(s, t, x, W)$, $s \leq t$, $x \in \mathbb{R}^d$.

$$1 + \gamma := \frac{\beta}{p_1} + \frac{1}{p_2}, \quad \gamma > 0.$$

Let us pick $\alpha \in (0, 1)$ such that

$$\frac{\alpha\beta}{p_1} + \frac{\alpha}{p_2} = 1 + \delta, \quad \delta > 0.$$

Let us estimate $|Y_r - X(u, r, Y_u, W)|$. It is clear that we have the following trivial bound:

$$\begin{aligned} |Y_r - X(u, r, Y_u, W)| &\leq \int_u^r |b(s, Y_s) - b(s, X(u, s, Y_u, W))| ds \leq \\ &\leq 2 \int_u^r M_1(s) ds \leq 2 \|M_1\|_{L^{q_1}[0, T]} |r - u|^{\frac{1}{p_1}} \end{aligned}$$

The previous estimate can be improved if we take into account the Hölder-continuity of the coefficient b :

$$\begin{aligned} |Y_r - X(u, r, Y_u, W)| &\leq \int_u^r |b(s, Y_s) - b(s, X(u, s, Y_u, W))| ds \leq \\ &\leq \int_u^r M_2(s) |Y_s - X(u, s, Y_u, W)|^\beta ds \leq K' \int_u^r M_2(s) |r - u|^{\frac{\beta}{p_1}} ds \leq \\ &\leq K' \|M_2\|_{L^{q_2}[0, T]} |r - u|^{\frac{\beta}{p_1} + \frac{1}{p_2}} = K' \|M_2\|_{L^{q_2}[0, T]} |r - u|^{1+\gamma}. \end{aligned}$$

Define sets $\{S_n\}$ as

$$S_n := \left\{ k/2^n; k \in \{0, 1, \dots, 2^n - 1\} \right\}, \quad |S_n| = 2^n$$

Using the property 4 from Proposition 2.3 with $\eta = 1$ and $S = \{S_n\}_{n=1}^\infty$ we obtain Ω' with $P(\Omega') = 1$ such that the following estimate holds:

$$|X(s, t, x, W) - X(s, t, y, W)| \leq C(\alpha, T, N, \omega) |x - y|^\alpha, \quad |x - y| \leq \frac{1}{2^n}, \quad s \in S_n$$

Now let us prove that for each trajectory $W \in \Omega'$ the equation 1 has exactly one solution. Let us choose a sufficiently large number K . Let $t \in S_{k'}$, where $k' \geq K$. Define an auxiliary function f by the formula

$$f(s) = X(s, t, Y_s, W) - X(0, t, x, W), \quad s \in [0, t].$$

Let $k \geq k'$ and u, r be of the form $\frac{i}{2^k}, \frac{i+1}{2^k}$ respectively, in particular $u, r \in S_k$. Recall that

$$|Y_r - X(u, r, Y_u, W)| \leq C |r - u|^{1+\gamma} \leq C 2^{-k\gamma} 2^{-k}$$

Since K is supposed to be sufficiently large we may assume that $C2^{-K\gamma} \leq 1$. Consequently,

$$|Y_r - X(u, r, Y_u, W)| \leq 2^{-k}$$

Then

$$\begin{aligned} |f(r) - f(u)| &= |X(r, t, Y_r, W) - X(u, t, Y_u, W)| = \\ &= |X(r, t, Y_r, W) - X(r, t, X(u, r, Y_u, W), W)| \leq \\ &\leq C(\alpha, S, T, M(x, W), \omega) |Y_r - X(u, r, Y_u, W)|^\alpha. \end{aligned}$$

Finally,

$$|f(r) - f(u)| \leq C(\alpha, S, T, M(x, W), \omega) |r - u|^{1+\delta}.$$

Due to the arbitrariness of k we conclude

$$f(t) = X(x, 0, t, W) - Y_t = 0.$$

Since t was an arbitrary dyadic number in $[0, 1]$ with a sufficiently large denominator, the continuity of Y_t and $X(x, 0, t, W)$ implies the equality $Y_t = X(x, 0, t, W)$ for each $t \in [0, 1]$. The proof is complete. \square

Now we show how to prove uniqueness in the case of a Borel measurable drift following [11]. Similarly to the proof of Theorem 3.1, it is readily seen that without loss of generality we can assume that $b(t, x) = b(t, x)I_{\{|x| < N\}}$ and $\|b\|_\infty \leq 1$.

Below we will need the following set of functions:

$$\begin{aligned} Lip_N([r, u], \mathbb{R}^d) &:= \\ &:= \{h \in C([r, u], \mathbb{R}^d) \mid |h(t) - h(s)| \leq |t - s|, s, t \in [r, u], \max_{s \in [r, u]} |h(s)| \leq N\} \end{aligned}$$

with the uniform metric $\varrho(h_1, h_2) = \|h_1 - h_2\|_\infty$.

The following result was proved in [11] and the corresponding arguments remain unchanged.

Lemma 3.6. *There exist constants $C, \zeta > 0$, independent of $l = u - r$, and a set Ω' such that*

$$P(\Omega \setminus \Omega') \leq C \exp(-l^{-\zeta})$$

and for any $h_1, h_2 \in Lip_N([r, u], \mathbb{R}^d)$ with $\|h_1 - h_2\|_\infty \leq 4l$, $W \in \Omega'$ the following inequality holds:

$$|\varphi(h_1, W) - \varphi(h_2, W)| \leq Cl^{\frac{4}{3}}.$$

We can now proceed to the proof of Theorem 1.1.

Proof. Let us fix a positive number N . Let C, ζ be constants found in Lemma 3.6. For each k we split the interval $[0, 1]$ into $M = 2^k$ closed subintervals

$$\left[0, \frac{1}{M}\right], \dots, \left[\frac{M-1}{M}, M\right].$$

Applying Lemma 3.6 to each interval $\left[\frac{i}{M}, \frac{i+1}{M}\right]$ we can find the corresponding sets $\Omega_{k,i}$. Let

$$\Omega_k := \bigcap_{i=0}^{M-1} \Omega_{k,i}.$$

With the help of the Borel–Cantelli lemma it is easy to show that the set

$$\Omega' := \liminf_{k \rightarrow \infty} \Omega_k = \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} \Omega_k$$

has probability 1.

Define S_n as

$$S_n := \left\{ k/2^n; k \in \{0, 1, \dots, 2^n - 1\} \right\}, \quad |S_n| = 2^n$$

Using the property 4 from Proposition 2.3 with $\eta = 1$ and $S = \{S_n\}_{n=1}^{\infty}$ we may assume (removing, if necessary, a set of zero probability) that on the set Ω' the following estimate holds:

$$|X(s, t, x, W) - X(s, t, y, W)| \leq C(\alpha, T, N, \omega) |x - y|^\alpha, \quad |x - y| \leq \frac{1}{2^n}, s \in S_n$$

Let us show that for each $W \in \Omega'$ such that

$$|x| + \max_{t \in [0, 1]} |W_t| + 1 \leq N,$$

the equation 1 has a unique solution. Indeed, let Y_t be a solution to the equation 1. It is not difficult to see that $|Y_t| \leq N$ for each $t \in [0, 1]$. Due to our choice of Ω' there exists $K = K(\omega)$ such that for all $k \geq K$ the Brownian trajectory W belongs to Ω_k . Let

$$M' = 2^{k'}, \quad r = \frac{i}{M'}, \quad \text{where } k' \geq K.$$

Let us define an auxiliary function f on the interval $[0, r]$ by the following formula:

$$f(t) := X(x, 0, r, W) - X(Y_t, t, r, W).$$

We observe that for any $s \leq t$, by the definition of a flow we have

$$\begin{aligned} f(t) - f(s) &= -X(Y_t, t, r, W) + X(Y_s, s, r, W) = \\ &= -X(Y_t, t, r, W) + X(X(Y_s, s, t, W), r, W). \end{aligned}$$

The difference $Y_t - X(Y_s, s, t, W)$ can be represented as follows:

$$\begin{aligned} Y_t - X(Y_s, s, t, W) &= \\ &= \int_s^t b\left(u, Y_s + W_u - W_s + \int_s^u b(r, Y_r) dr\right) du - \\ &\quad \int_s^t b\left(u, Y_s + W_u - W_s + \int_s^u b(r, X_r) dr\right) du = \\ &= \int_s^t b(u, W_u + h_1(u)) du - \int_s^t b(u, W_u + h_2(u)) du, \end{aligned}$$

where

$$h_1(u) = Y_s - W_s + \int_s^u b(r, Y_r) dr, \quad h_2(u) = Y_s - W_s + \int_s^u b(r, X_r) dr.$$

Let $k \geq k'$ $M = 2^k$. If we take s, t of the form $\frac{i}{M}$ and $\frac{i+1}{M}$, respectively, then we obtain the following estimate:

$$|Y_t - X(Y_s, s, t, W)| \leq \frac{C}{M^{\frac{4}{3}}}$$

Since we may assume that M is sufficiently large this inequality implies the bound

$$|Y_t - X(Y_s, s, t, W)| \leq \frac{1}{M}.$$

Hence there exists a positive constant $C = C(N, S, W)$ such that

$$|f(t) - f(s)| \leq C|Y_t - X(Y_s, s, t, W)|^{\frac{4}{5}}.$$

$$\left| f\left(\frac{i+1}{M}\right) - f\left(\frac{i}{M}\right) \right| \leq \left(\frac{C}{M^{\frac{4}{3}}} \right)^{\frac{4}{5}},$$

and consequently

$$|f(r)| \leq \frac{C}{M^{\frac{1}{15}}}.$$

Due to the arbitrariness of k we conclude

$$f(r) = X(x, 0, r, W) - Y_r = 0.$$

Since r was an arbitrary dyadic number in $[0, 1]$ with a sufficiently large denominator, the continuity of Y_t and $X(x, 0, t, W)$ implies the equality $Y_t = X(x, 0, t, W)$ for each $t \in [0, 1]$. The proof is complete. \square

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REFERENCES

- [1] Davie A.M. Uniqueness of solutions of stochastic differential equations. International Mathematics Research Notices, 2007, V. 2007.
- [2] Fedrizzi E., Flandoli F. Hölder flow and differentiability for SDEs with nonregular drift. Stoch. Anal. Appl., 2013, V. 31, N4, P. 708–736.
- [3] Fedrizzi E., Flandoli F. Pathwise uniqueness and continuous dependence for SDEs with nonregular drift. arXiv preprint arXiv:1004.3485, 2010.
- [4] Van Kampen E.R. Remarks on systems of ordinary differential equations. Amer. J. Math., 1937, V. 59, N1, P. 144–152.
- [5] Flandoli F. Regularizing properties of Brownian paths and a result of Davie. Stochastics and Dynamics, 2011, V. 11, N02n03, P. 323–331.
- [6] Föllmer H., Protter P., Shiryaev A.N. Quadratic covariation and an extension of Ito's formula. Bernoulli, 1995, V.1, N1-2, P. 149–169.
- [7] Kolmogorov A.N., Tikhomirov V. M. ε -entropy and ε -capacity of sets in function spaces. Uspekhi Matem. Nauk, 1959, V. 14, N2, P. 3–86 (in Russian). English translation: Amer. Math. Soc. Transl. Ser. 2, 1961, V. 17, P. 277–364
- [8] Krylov N.V., Röckner M. Strong solutions of stochastic equations with singular time dependent drift. Probab. Theory Related Fields, 2005, V. 131, N2, P. 154–196.
- [9] Kallenberg O. Foundations of modern probability. Probability and its Applications. Springer-Verlag, New York, II edition, 2002
- [10] Priola E. Davie's type uniqueness for a class of SDEs with jumps. preprint arXiv:1509.07448 2015.
- [11] Shaposhnikov A.V. Some remarks on Davie's uniqueness theorem. preprint arXiv:1401.5455 2014.